MATHEMATICS

ON THE BEST APPROXIMATION PROPERTY IN NON-ARCHIMEDEAN NORMED SPACES

BY

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1. *Introduction*

Let E be a non-archimedean normed space over a field k with non-trivial non-archimedean valuation which is supposed to be complete in the topology derived from the valuation. Let $V \subset E$ be a closed linear subspace of E. For a given $x \in E$, a best approximation of x in V is defined to be an element $\xi \in V$ such that

$$\|x-\xi\| = \inf_{g \in V} \|x-g\|.$$

It has been shown by A. F. MONNA [3] that a best approximation of an element $x \in E$ in V when it exists, is never uniquely determined unless $V = \{0\}$. Concerning the existence of best approximations A. F. MONNA [3, problem 1] posed the following problem.

Let E be a non-archimedean normed space satisfying the following condition.

(*) Let V be a closed subspace of E. Let $x_0 \in E$, $x_0 \notin V$. Suppose a best approximation of x_0 in V exists. Does every $x \in E$ have a best approximation in V?

The purpose of this note is to give a sufficient condition for the existence of best approximations of every element of E in V. The condition (*) is replaced by the following condition:

(**) For every closed proper subspace V of E there exists at least one element $x \in E$, $x \notin V$ which has a best approximation in V.

We show under the condition (**) that for any closed subspace V of E every element in E has a best approximation in V. That the condition (*) is not sufficient for the existence of best approximations of every element of E in a fixed closed subspace V is shown by a counter-example.

2. *Our first aim is to prove the following theorem*

Theorem. Let E be a non-archimedean normed space over k satisfying the condition (**). Then for any closed subspace $V \subset E$ every element in E has a best approximation in V.

For the proof of this theorem we need two lemmas.

Lemma 1. Let V be a closed subspace of E. If $x_0 \in E$, $x_0 \notin V$ has a best approximation in V, then every element in the space $\{x_0, V\}$ has a best approximation in V.

Proof. Clear.

Lemma 2. Let V, W be two subspaces of E, $V \subset W$. If $x \in E$ has a best approximation in W and if every element in W has a best approximation in V, then x has a best approximation in V.

Proof. By assumption, there is an element $\xi \in W$ such that $\|x-\xi\| = \inf_{y \in W} \|x-y\|$. For this ξ, there is an element $\xi_0 \in V$ such that $\|\xi-\xi_0\| = \inf_{z \in V} \|\xi - z\|$. We claim that $\|x-\xi_0\| = \inf_{z \in V} \|x-z\|$. First $\|x-\xi_0\| \geqq \inf_{z \in V} \|x-z\|$ since $\xi_0 \in V$. Furthermore $\|x-\xi_0\| \leqq \max \{\|x-\xi\|, \|\xi-\xi_0\|\}$. Since $\|x-\xi\| = \inf_{y \in W} \|x-y\| \leqq \inf_{z \in V} \|x-z\|$, we have $\|x-\xi\| \leqq \|x-\xi_0\|$. Now if $\|x-\xi\| = \|x-\xi_0\|$, then $\|x-\xi\| = \inf_{y \in W} \|x-y\| \leqq \inf_{z \in V} \|x-z\|$ and we are done. Thus, suppose that $\|x-\xi\| < \|x-\xi_0\|$, then $\|x-\xi_0\| = \|\xi-\xi_0\|$. Since $\|\xi - z\| \leqq \max \{\|x-\xi\|, \|x-z\|\}$ and $\|x-\xi\| = \inf_{y \in W} \|x-y\| \leqq \|x-z\|$ for all $z \in V$, we have $\|\xi-z\| \leqq \|x-z\|$. This implies that $\|x-\xi_0\| = \|\xi-\xi_0\| \leqq \inf_{z \in V} \|x-z\|$ which completes the proof.

Now we prove the theorem by using Zorn's Lemma.

Let $\mathscr{F}$ be the family of all subspaces $U \supset V$ such that every element of U has a best approximation in V. $\mathscr{F}$ is not empty and it is an inductive family with respect to the inclusion. In fact if $\{U_\alpha\}$ is totaly ordered, then $U = \bigcup_\alpha U_\alpha$ is clearly in $\mathscr{F}$. Let W be a maximal element in $\mathscr{F}$. We claim that W is closed. Let $x \in \overline{W}$ (closure of W) and let $x' \in W$ with:

$$\|x-x'\| < \inf_{y \in V} \|x-y\| \leqq \|x-y\|.$$

For $x' \in W$ there exists $\xi_{x'} \in V$ such that

$$\|x'-\xi_{x'}\| = \inf_{y \in V} \|x'-y\|.$$

But $\inf_{y \in V} \|x-y\| \leqq \|x-\xi_{x'}\| \leqq \max \{\|x-x'\|, \|x'-\xi_{x'}\|\}$. Therefore $\|x-x'\|$ can not be equal to $\|x'-\xi_{x'}\|$. Infact $\|x-x'\| < \|x'-\xi_{x'}\|$. Therefore $\|x-\xi_{x'}\| = \|x'-\xi_{x'}\| = \inf_{y \in V} \|x'-y\|$. Now $\|x'-y\| \leqq \max \{\|x-y\|, \|x-x'\|\}$. But for all $y \in V$, $\|x-y\| > \|x-x'\|$. Therefore $\|x'-y\| = \|x-y\|$. Hence

$$\|x-\xi_{x'}\| = \inf_{y \in V} \|x-y\|.$$

This implies that W is closed.

Now if $W \neq E$, then by condition (**) there exists an $x \notin W$ such that

x has a best approximation in W. Since every element in W has a best approximation in V so, by lemma 2, x has a best approximation in V. Then, by lemma 1, every element in the subspace $\{x, W\}$ has a best approximation in V. Thus $\{x, W\} \in \mathscr{F}$. This contradicts the maximality of W and the proof is complete.

3. *Counterexample*

We will show by an example that the condition (*) is not sufficient. We will consider the example given by INGLETON [1].

Let E be the space of all formal power series such that every non-zero element $x \in E$ is of the form

$$x = a_1 t^{\alpha_1} + a_2 t^{\alpha_2} + \dots$$

where the α_i are rational numbers well-ordered in natural (ascending) order, and the a_i are non-zero coefficients taken from some field Γ. E is a non-archimedean valued field with $\|x\| = e^{-\alpha_1}$ $(a_1 \neq 0)$ and may be regarded as a non-archimedean space over the subfield K with elements such that $\{\alpha_i\}$ is a finite set or simple sequence tending to infinity. K is a closed subfield of E. K is complete but it is not spherically complete [2].

Let u_0 be an element of E, $u_0 \notin K$ so that $u_0 = a_1 t^{\alpha_1} + a_2 t^{\alpha_2} + \dots$, where $\alpha_1 < \alpha_2 < \dots$ is a sequence which converges to an irrational number ϱ. Then for $x \in K$, $\|u_0 - x\| = e^{-\alpha_{n+1}}$ for some n and $\inf_{x \in K} \|u_0 - x\| = e^{-\varrho}$. But, there is no element $\xi \in K$ so that $\|u_0 - \xi\| = e^{-\varrho}$. Hence u_0 has no best approximation in K.

Let u_1 be an element of $E \setminus K$ such that $\|u_1\| = \alpha \in \|K\|$. Consider the vector space

$$W = K + Ku_0 + Ku_1.$$

Let us define a norm $\|\cdot\|^*$ on W as follows:

For $y \in W$, $y = x + \lambda u_1$ where $x \in K + Ku_0$, $\lambda \in K$

$$\|y\|^* = \|x + \lambda u_1\|^* := \max \{\|x\|, \|\lambda\|\alpha\}.$$

This norm is in fact a non-archimedean norm. It suffices to show that $\|\cdot\|^*$ is ultrametric:

Let $x_1 + \lambda_1 u_1$, $x_2 + \lambda_2 u_1$ be two elements of W. Then

$$\begin{aligned}
\|x_1 + \lambda_1 u_1 + x_2 + \lambda_2 u_1\|^* &= \|x_1 + x_2 + (\lambda_1 + \lambda_2) u_1\|^* \\
&= \max \{\|x_1 + x_2\|, \|\lambda_1 + \lambda_2\|\alpha\} \\
&\leq \max \{\max (\|x_1\|, \|x_2\|), \max (\alpha\|\lambda_1\|, \|\lambda_2\|\alpha)\} \\
&= \max \{\max (\|x_1\|, \|\lambda_1\|\alpha), \max (\|x_2\|, \|\lambda_2\|\alpha)\} \\
&= \max (\|x_1 + \lambda_1 u_1\|^*, \|x_2 + \lambda_2 u_1\|^*).
\end{aligned}$$

This shows that W is a non-archimedean normed space over K.

Now we show that u_1 has a best approximation in K. For this, consider $\inf_{y \in K} \|u_1 - y\|^* = \inf_{y \in K} \{\max(\|u_1\|, \|y\|)\} = \alpha$. Let $y_0 \in K$ such that $\|y_0\| < \alpha$, then $\|u_1 - y_0\|^* = \alpha$. Therefore u_1 has a best approximation in K. This shows that $u_1 \in W \backslash K$ has a best approximation in K but not every element of W has a best approximation in K.

Remark. It is worthwhile to mention that the set of vectors in a non-archimedean normed space E which have best approximations in a fixed subspace V of E is not, in general, a vector space. Indeed, consider the spaces W and K given above. Let z be an element of W of the form $z = \lambda u_0 - u_1$ where u_0, u_1 are the same as mentioned above, and $\lambda \in K$ is chosen so that $\|\lambda u_0\| < \alpha$, $(\alpha = \|u_1\|)$. Clearly z has a best approximation in K but $z + u_1$ does not have a best approximation in K.

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